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Large strain contact of a rubber wedge with a rigid notch

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Abstract

The contact problem of a rigid notch with a rubber wedge is analyzed by asymptotic method. The stress and strain field near the apex of the rubber wedge is shown to be singular, but the angular distributions of quantities are uniform. The analytical conclusion is verified by the results of finite element calculation.

Keywords: Rubber wedge; Rigid notch; Contact; Large strain

1. Introduction

The earliest solution to contact problem was given by Hertz (1881) for two spherical bodies, based on linear elastic theory. When the contact surfaces contain vertex, the problem cannot be solved in the framework of infinitesimal strain theory. In Gao and Gao (2000), the vertex contact problem was solved based on the elastic law given by Gao (1997). The result in Gao and Gao (2000) shows that there are two shrinking sectors and one expanding sector surrounding the apex of rigid wedge. When the angle of rigid wedge tends to π the solution becomes invalid and the singularity disappears. The smooth contact problem of a rubber wedge with a rigid notch is investigated in the present paper. The deformation pattern and analytical result of this paper are quite different from that case analyzed by Gao and Gao (2000).

2. Basic equation

Consider a three dimensional domain of rubber material, let **P** and **Q** denote the position vectors of a point before and after deformation, respectively. $x^i (i = 1, 2, 3)$ is the Lagrangian coordinate. Two set of local triads can be defined as

$$\mathbf{P}_i = \frac{\partial \mathbf{P}}{\partial x^i}, \qquad \mathbf{Q}_i = \frac{\partial \mathbf{Q}}{\partial x^i} \tag{1}$$

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Three independent invariants can be introduced,

$$I = (\mathbf{P}^i \cdot \mathbf{P}^j) \cdot (\mathbf{Q}_i \cdot \mathbf{Q}_j), \qquad I_{-1} = (\mathbf{P}_i \cdot \mathbf{P}_j) \cdot (\mathbf{Q}^i \cdot \mathbf{Q}^j)$$
$$J = V_Q / V_P \tag{2}$$

in which summation rule is implied. $\mathbf{P}^i$ and $\mathbf{Q}^i$ are the conjugates of $\mathbf{P}_i$ and $\mathbf{Q}_i$ respectively, $V_* = (*_1, *_2, *_3)$ with ($*_1$, $*_2$, $*_3$) being the mixed product of vector $*_1$, $*_2$, and $*_3$, Let W denote the strain energy per undeformed unit volume, then the Cauchy stress can be expressed as

$$\tau = J^{-1} \frac{\partial W}{\partial \mathbf{Q}_i} \otimes \mathbf{Q}_i \tag{3}$$

where $\otimes$ is the dyadic symbol.

Gao (1997) gave a strain energy form

$$W = a(I^n + I_{-1}^n) \tag{4}$$

where a and n are material constants.

From Eqs. (3) and (4) it follows that

$$\tau = 2naJ^{-1}(I^{n-1}\mathbf{d} - I_{-1}^{n-1}\mathbf{d}^{-1}) \tag{5}$$

where

$$\mathbf{d} = (\mathbf{P}^i \cdot \mathbf{P}^j)\mathbf{Q}_i \otimes \mathbf{Q}_j, \qquad \mathbf{d}^{-1} = (\mathbf{P}_i \cdot \mathbf{P}_j)\mathbf{Q}^i \otimes \mathbf{Q}^j \tag{6}$$

The material behavior according to Eq. (5) was discussed in detail by Gao and Gao (1999). For small strain case, let

$$\varepsilon = \tfrac{1}{2}(\mathbf{d} - \mathbf{U}) \tag{7}$$

$\mathbf{U}$ is unit tensor

$$\mathbf{U} = \mathbf{P}_i \otimes \mathbf{P}^i = \mathbf{Q}_i \otimes \mathbf{Q}^i \tag{8}$$

then Eq. (5) is reduced to

$$\tau = 8na3^{n-1}[\varepsilon - \tfrac{1}{3}(n-1)i_1\mathbf{U}] \tag{9}$$

in which

$$i_1 = \varepsilon : \mathbf{U} \tag{10}$$

Evidently, when $\varepsilon \to 0$, τ vanishes. Besides, Young's modulus and Poisson's ratio are

$$E = \frac{8n^2}{2n+1} 3^n a, \qquad v = \frac{n-1}{2n+1} \tag{11}$$

The equilibrium equation can be written as

$$\frac{\partial}{\partial x^i}(V_Q \tau \cdot \mathbf{Q}^i) = 0 \tag{12}$$

3. Asymptotic equations

Shown in Fig. 1(a) and (b) are the cross-section of a rigid notch with a rubber wedge before and after deformation respectively. $2A$ and $2B$ denote the angles of rigid notch and rubber wedge.

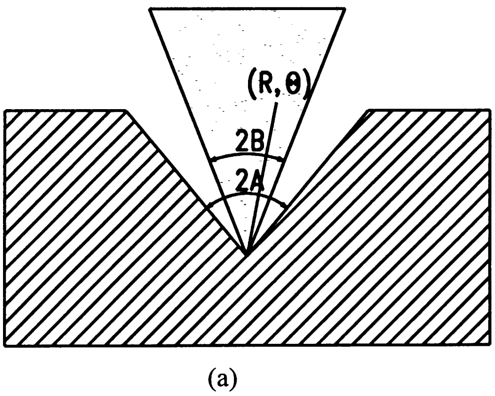

Fig. 1. Contact of a rubber wedge with a rigid notch: (a) before deformation, (b) after deformation.

In order to describe the deformation, two Lagrangian coordinates are taken such that (R, Θ, Z) is cylindrical coordinate in undeformed configuration while (r, θ, z) is cylindrical coordinate in deformed configuration, the axes are along the notch valley. The problem can be treated as plane strain case, so $z \equiv Z$, then the deformation can be described by the mapping from (r, θ) to (R, Θ), which is assumed to be

$$R = r^{1-\delta}\varphi(\theta), \qquad \Theta = \psi(\theta) \tag{13}$$

where δ is a positive exponent to be determined. The first mapping function of Eq. (13) possesses singularity but the second is a regular function. Therefore the domain of rubber wedge does not contain shrinking sector or expanding sector, this is unlike the case analyzed by Gao and Gao (2000).

The deformation near the wedge apex can also be described by the following mapping functions:

$$r = R^{1+\beta}f(\Theta), \qquad \theta = g(\Theta) \tag{14}$$

where

$$\beta = \frac{\delta}{1-\delta} \tag{15}$$

functions f and g can be expressed by φ and ψ. In the following, only Eq. (13) is discussed.

Let $\mathbf{e}_R$, $\mathbf{e}_\Theta$, $\mathbf{e}_r$, $\mathbf{e}_\theta$, denote the unit vectors in (R, Θ) and (r, θ) systems respectively, i.e.

$$\mathbf{e}_R = \frac{\partial \mathbf{P}}{\partial R}, \qquad \mathbf{e}_\Theta = \frac{1}{R}\frac{\partial \mathbf{P}}{\partial \Theta} \tag{16}$$

$$\mathbf{e}_r = \frac{\partial \mathbf{Q}}{\partial r}, \qquad \mathbf{e}_\theta = \frac{1}{r}\frac{\partial \mathbf{Q}}{\partial \theta} \tag{17}$$

According to Eqs. (13) and (16) it follows that

$$\begin{cases} \mathbf{P}_r = \frac{\partial \mathbf{P}}{\partial r} = r^{-\delta}(1-\delta)\varphi\mathbf{e}_R \\ \mathbf{P}_\theta = \frac{\partial \mathbf{P}}{\partial \theta} = r^{1-\delta}(\varphi'\mathbf{e}_R + \varphi\psi'\mathbf{e}_\Theta) \end{cases} \tag{18}$$

From Eq. (18), it follows that

$$\begin{cases} \mathbf{P}^r = r^{\delta}q^{-1}(\varphi\psi'\mathbf{e}_R - \varphi'\mathbf{e}_\Theta) \\ \mathbf{P}^\theta = r^{\delta-1}q^{-1}(1-\delta)\varphi\mathbf{e}_\Theta \end{cases} \tag{19}$$

in which

$$q = (1-\delta)\varphi^2\psi' \tag{20}$$

Using Eqs. (18), (19) and (2), the invariants can be obtained,

$$I = r^{2\delta}q^{-2}p, \qquad I_{-1} = r^{-2\delta}p, \qquad J = r^{2\delta}q^{-1} \tag{21}$$

where

$$p = \varphi'^2 + (1-\delta)^2\varphi^2 + \varphi^2\psi'^2 \tag{22}$$

Eq. (21) shows that I can be neglected comparing with I_{-1} in the strain energy W, so that

$$W = aI_{-1}^n \tag{23}$$

then Eqs. (5), (6) and (18) are combined to give

$$\tau = -2nar^{-2(n+1)\delta}qp^{n-1}\Big[(1-\delta)^2\varphi^2\mathbf{e}_r\otimes\mathbf{e}_r + (\varphi'^2+\varphi^2\psi'^2)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + (1-\delta)\varphi\varphi'(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r)\Big] \tag{24}$$

Noting that

$$V_Q = \mid Q_r \times Q_\theta \mid = r \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (12), it follows that

$$\begin{cases} \varphi''\left[1+\frac{2}{p}(n-1)\varphi'^2\right] + \frac{\psi''\varphi'}{\psi'}\left[1+\frac{2}{p}(n-1)\varphi^2\psi'^2\right] + \left(2n-\frac{\delta}{1-\delta}\right)\frac{\varphi'^2}{\varphi} - 2(n-1)\frac{\varphi'^4}{\varphi p} \\ \qquad -\frac{\varphi}{1-\delta}\left\{\psi'^2 - (1-\delta)^2[1-2(n+1)\delta]\right\} = 0 \\ \varphi''\left(\frac{\varphi'^2}{\varphi^2\psi'^2}-1\right) + 2\frac{\varphi'\psi''}{\psi'} + \left(1+\frac{2}{1-\delta}\right)\frac{\varphi'^2}{\varphi} + \frac{\varphi'^2}{\psi'^2\varphi}\left(1-\delta+\frac{\delta}{1-\delta}\frac{\varphi'^2}{\varphi^2}\right) \\ \qquad +\frac{\varphi}{1-\delta}\left\{\psi'^2-(1-\delta)^2[1-2(n+1)\delta]\right\} = 0 \end{cases} \tag{26}$$

4. Boundary conditions and solution

The asymptotic analysis (AA) of wedge tip field is only given for small value of r, therefore the governing equations (26) only contain variable θ. The unilateral displacement conditions for contact problem are

$$\psi(A) = B, \qquad \psi(-A) = -B \tag{27}$$

The traction conditions for smooth contact are

$$\tau^{r\theta} = 0 \quad \text{at } \theta = \pm\text{A} \tag{28}$$

According to Eqs. (24) and (28), is follows that

$$\varphi'(A) = \varphi'(-A) = 0 \tag{29}$$

It should be mentioned that discussed in this paper is only the asymptotic behavior of the wedge tip, i.e. near the apex of the wedge. The solution will be restricted only to the contact area. Outside of this area, the boundary conditions must be changed to traction free conditions.

Eq. (26) under boundary conditions (27) and (29) can be solved numerically. The calculation result shows that the only solution is

$$\varphi = \varphi_0, \qquad \psi = \frac{B}{A}\theta \tag{30}$$

where φ_0 is a constant to indicate the amplitude of the field. Parameter φ_0 depends on the loading. The eigenvalue δ is given by

$$(1-\delta)^2[1-2(n+1)\delta] = \frac{B^2}{A^2} \tag{31}$$

Solutions (30) and (31) are not obtained by analysis, however, when these are substituted into Eq. (26), the equation is satisfied. From Eqs. (13) and (30), it follows that

$$R = \varphi_0 r^{1-\delta}, \qquad \Theta = \frac{B}{A}\theta \tag{32}$$

then Eq. (24) gives

$$\tau = -2nar^{-2(n+1)\delta}C(1+C^2)^{n-1}[\varphi_0(1-\delta)]^{2(n+1)}(\mathbf{e}_r \otimes \mathbf{e}_r + C^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \tag{33}$$

or

$$\tau = -2na\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^{2(n+1)} C(1+C^2)^{n-1}(\mathbf{e}_r \otimes \mathbf{e}_r + C^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \tag{34}$$

where

$$C = [1-2(n+1)\delta]^{1/2} \tag{35}$$

So, the completely analytical solution for the asymptotic behavior of the wedge tip is obtained.

5. Finite element result

The contact problem is also calculated by finite element using total Lagrangian (TL) method. We take $a = 1$, $n = 2$. The mesh division is shown in Fig. 2. Isoparametric element with four nodes is used, and 500 elements with 551 nodes are taken. The minimum highness of an element is 2×10^{-5} times of the wedge height. For the case of $A = 50°$, $B = 30°$, the relation of r–R and τ–r obtained by finite elements and theory are plotted in Fig. 3(a) and (b) respectively. For the case of $A = 70°$, $B = 15°$, the results are plotted in Fig. 4(a) and (b). While $R = 0.16 \times 10^{-4}$, the angular distribution of τ^{rr}, $\tau^{\theta\theta}$ are plotted in Fig. 5, for $A = 70°$, $B = 15°$, the discrepancy between theory and calculation is less than 3.5%. The angular distribution of τ^{rr}, $\tau^{\theta\theta}$ for $A = 50°$, $B = 30°$ are not plotted, the discrepancies between theoretical and calculated τ^{rr}, $\tau^{\theta\theta}$ are less than 0.6‰. The shape of deformed wedge is shown in Fig. 6.

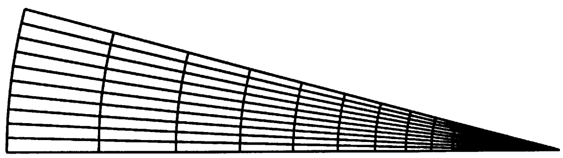

Fig. 2. FEM mesh.

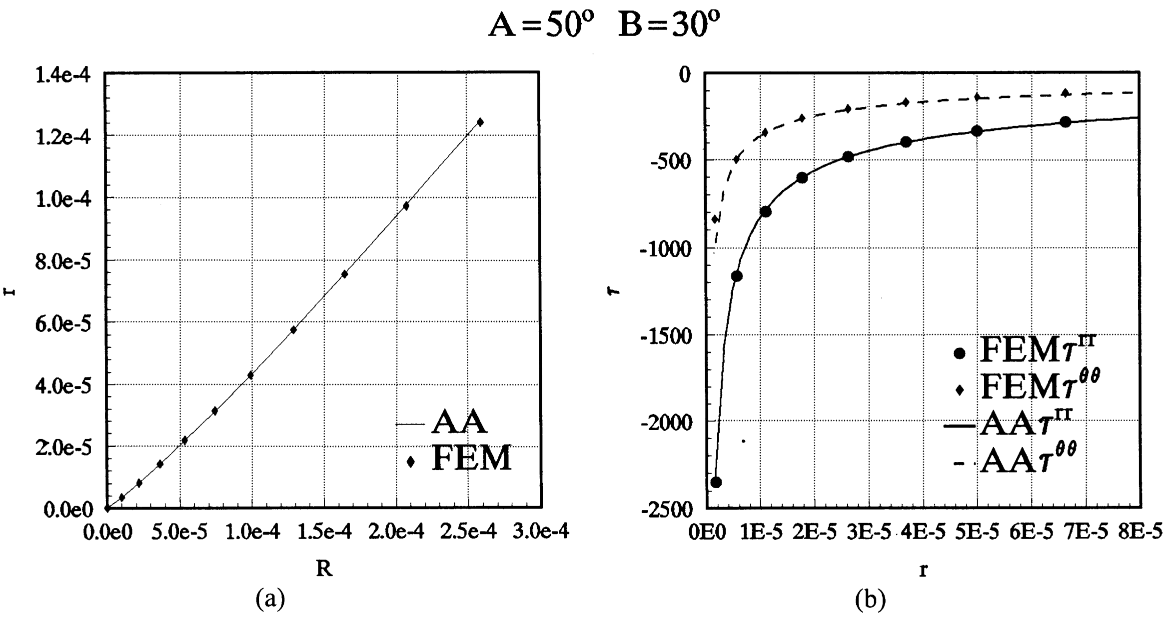

Fig. 3. Results of AA and FEM for $A = 50°$, $B = 30°$: (a) the curves of r–R, (b) the curves of τ–r.

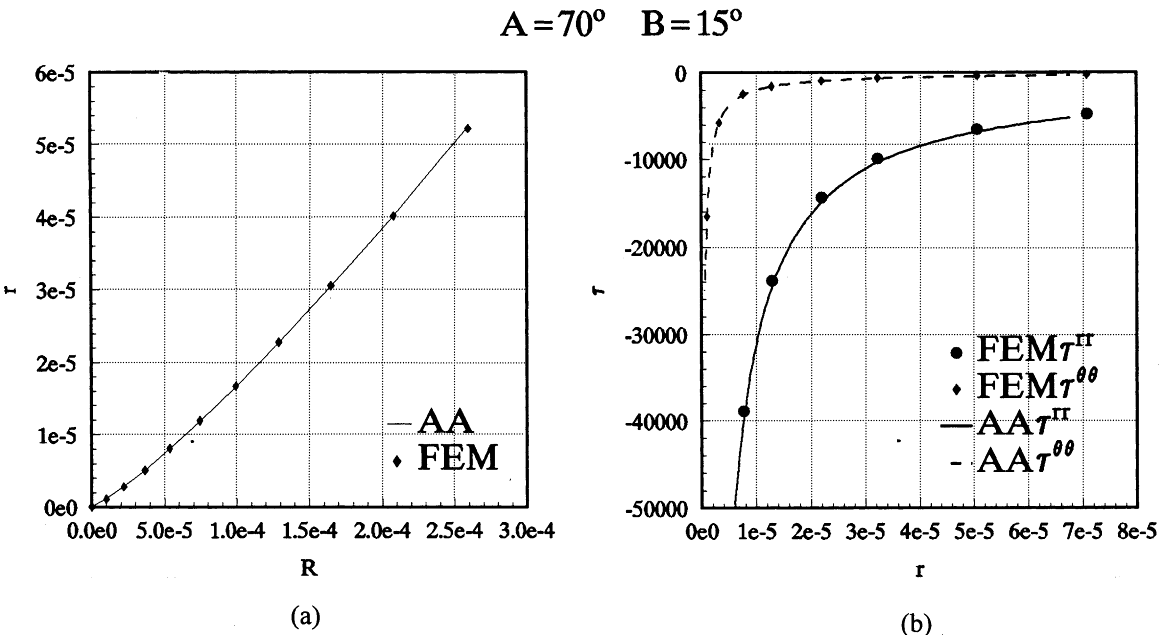

Fig. 4. AA and FEM for $A = 70°$, $B = 15°$: (a) the curves of r–R, (b) the curves of τ–r.

The extremely close correspondence of the analysis with the result of FEM can be explained as follows: (1) The elastic law is typical but simple. (2) The deformation pattern near the wedge apex can be well described by the mapping function, i.e. the neglected higher order terms are indeed small.

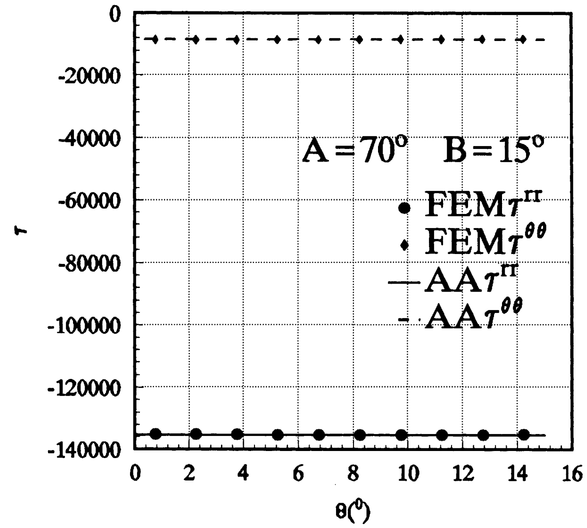

Fig. 5. Curves of τ–θ from AA and FEM for $A = 70°$, $B = 15°$.

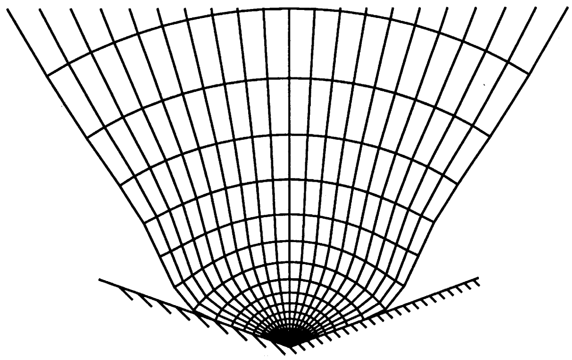

Fig. 6. The shapes of wedge after deformation for $A = 70°$, $B = 15°$.

6. Conclusions

The analytically asymptotic solution to the contact problem discussed in this paper is obtained and verified by FEM calculation. The contact problem of a rigid notch with a rubber wedge is unlike the contact problem with a rigid wedge. The wedge tip field obtained in this paper does not contain either shrinking sector or expanding sector. The ratio of Θ to θ is constant near the wedge tip.

The mapping function for radius is singular, the singularity δ depends on the ratio of angles B/A. Figs. 3(a) and 4(a) reflect the mapping function (13) very well. When the wedge tip is approached, stress τ^{rr} and $\tau^{\theta\theta}$ possess singularity of order $r^{-2(n+1)\delta}$. The angular distribution of τ^{rr} and $\tau^{\theta\theta}$ are constants. This can be obtained by analysis and also can be seen from the curves of Fig. 5. As for the radial variation of τ^{rr} and $\tau^{\theta\theta}$, Figs. 3(a) and 4(a) shown the evident singularity, when $r \to 0$, they tend to $-\infty$.

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